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The geometry of dual isomonodromic deformations

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Abstract

The JMMS equations are studied using the geometry of the spectral curve of a pair of dual systems. It is shown that the equations can be represented as time-independent Hamiltonian flows on a Jacobian bundle.

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1. Introduction

In this paper we revisit the geometry of dual isomonodromic deformations of a linear system on $\mathbb{CP}_1$. The duality was originally studied, at least in the form in which we are interested, by Harnad [6]—although it is closely related to the 'Laplace transform' in the theory of Frobenius manifolds [2,3].

The problem is to construct isomonodromic deformations of a linear meromorphic differential operator on the Riemann sphere $\mathbb{CP}_1$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}w} - b - \sum_{i=1}^{n} \frac{N_i}{w - a_i}, \tag{1}$$

where the N_i's are rank one, $r \times r$ matrices, and $b = \mathrm{diag}(b_1, \ldots, b_r)$, with the b_i distinct. The corresponding linear system of differential equations has n regular singularities at the

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points a_i, together with an irregular singularity of Poincaré rank one at infinity, where the leading coefficient has distinct eigenvalues.

The isomonodromic deformations are governed by the JMMS equations [8]. These appear, or are closely related to equations that do appear, in many areas of mathematical physics, from impenetrable bose gases, where they were originally introduced, to Frobenius manifolds and Seiberg-Witten problems (see [3,5]).

We shall consider deformations that fix the position of the pole at infinity, but change the parameters b_i and a_i. Such deformations are equivalent to the deformations of a second 'dual system' of the same type, in which the roles of the two sets of parameters a_i and b_i are interchanged [6]. The dual system is constructed by using the rank-one property of the N_i's to write the first linear operator in the form

$$\frac{\mathrm{d}}{\mathrm{d}w} - G^{\mathrm{t}}(a - wI_n)^{-1}F + b, \tag{2}$$

where G, F are $n \times r$ matrices and $a = \mathrm{diag}(a_1, \dots, a_n)$. Then the dual operator is

$$\frac{\mathrm{d}}{\mathrm{d}z} - F(b - zI_r)^{-1}G^{\mathrm{t}} + a. \tag{3}$$

At a formal level, the duality is given by writing the first linear system of ODEs in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}w} + b\right) v - G^{\mathrm{t}}u = 0, \qquad (a - w)u + Fv = 0,$$

where v is an r-vector and u is an n-vector—this reduces to (2) on elimination of u. Then the second system is given by a formal application of the Laplace transform, to replace $\mathrm{d}/\mathrm{d}w$ by z, and z by $-\mathrm{d}/\mathrm{d}w$, as was done in a related context by Balser et al. [2] and later in the context of Frobenius manifolds by Dubrovin [4].

In his original investigation, Harnad considered Hamiltonian flows on certain subspaces of the loop algebra $\widetilde{\mathrm{gl}}(n)_-$, the subspaces being given by varying the term $F(b-zI_r)^{-1}G^{\mathrm{t}}$ in (3). He showed that they were generated by the polynomials invariant by the adjoint action of the loop group $\widetilde{\mathrm{Gl}}(n)$. By writing down the explicit Lax equations, he deduced that they correspond to the isomonodromic deformations of the system that change the position of the regular singularities. He showed that they could be viewed as time-dependent Hamiltonian flows on the symplectic manifold of pairs of $n \times r$ matrices, with respect to a canonical symplectic form. The 'times' are the positions of the poles, and the Hamiltonians are

$$H_i(\mathcal{N}) = \frac{1}{4\pi i} \oint \mathrm{tr}(\mathcal{N}(w, a))^2 \,\mathrm{d}w, \tag{4}$$

where

$$\mathcal{N}(w, a) = G^{\mathrm{T}}(a - wI_n)^{-1}F + b$$

and the integral is around a loop containing only the ith pole. They generate the nonautonomous system

$$\frac{\partial N_j}{\partial a_i} = \frac{[N_j, N_i]}{a_j - a_i}, \quad j \neq i, \quad i, j = 1, \dots, n, \qquad \frac{\partial N_i}{\partial a_i} = \left[b + \sum_{j\neq i} \frac{N_j}{a_j - a_i}, N_i \right].$$

Here, we shall show that the Hamiltonian theory can be understood in a different way by introducing natural variables conjugate to the a_i's and b_i's. We associate with the linear system (1) a *spectral curve* Σ in $\mathbb{CP}_1 \times \mathbb{CP}_1$, given by

$$\det(\mathcal{N}(a, w) - zI_r) = 0$$

and a line bundle $L \to \Sigma$ of degree nr (essentially the line bundle determined by the 'divisor coordinates' in [1]). The spectral curve and the line bundle for the system and its dual are the same (apart from interchanging the two copies of $\mathbb{CP}_1$).

The symplectic manifold $\mathcal{M}$ on which the flows are defined consists of triples (Σ, B, s), where B is a line bundle of degree zero over Σ and s is a section defined in a neighbourhood of the points $w = \infty$ and $z = \infty$. Two such sections are regarded as equivalent if the first two terms in their Taylor expansion of s at these points are the same, so the dimension of $\mathcal{M}$ is $2(nr + n + r)$.

There is a standard line bundle over $\mathbb{CP}_1 \times \mathbb{CP}_1$ (less (∞, ∞)) with transition function $\exp(wz)$: this encodes the behaviour of the differential operators at their singularities. The Hamiltonians h_i are labelled by the points at infinity, and they are in involution. They are defined by associating a meromorphic section of $B \otimes E$ with each point at infinity. The zeros q_i of the section are the 'divisor coordinates' in [1], and the Hamiltonian is given by a residue at infinity constructed from the section.

The two operators are recovered by treating the coordinates w and z as multiplication operators on the sections of a line bundle L of degree nr, which has divisor made up of the q_i's, together with points at infinity. Under a Hamiltonian flow, the q_i's move and L is twisted over the point at infinity associated with the Hamiltonian. The role of E here is critical: it gives rise to the twist, which can be seen as having its origin in the evolution of the diagonal exponentials in the formal solutions of the linear systems in [9]. Harnad's nonautonomous Hamiltonian description is the Marsden–Weinstein reduction by the Hamiltonian action.

2. The spectral curve

The relationship between the two systems is driven by the geometry of their common spectral curve $\Sigma \subset \mathbb{CP}_1 \times \mathbb{CP}_1$. This has equation

$$\det(zI_r - G^{\mathrm{T}}(a - wI_n)^{-1}F + b) = 0, \tag{5}$$

where w and z are the coordinates on the two copies of $\mathbb{CP}_1$. It is an n-fold cover of the z-copy of $\mathbb{CP}_1$, and an r-fold cover of the w-copy. For generic values of F and G, which we shall assume, it is a smooth curve. The dual problem gives the same curve (with the two coordinates interchanged). This follows either by using the identity

$$\begin{aligned}&\det(a - wI_n)\det(zI_r - G^{\mathrm{T}}(a - wI_n)^{-1}F + b)\\&\quad = \det(b - zI_r)\det(wI_n - F(b - zI_r)^{-1}G^{\mathrm{t}} + a),\end{aligned} \tag{6}$$

from [6]; or by noting that the curve is the set $\{(w, z)\}$ on which the linear equations

$$(a - w)u - Fv = 0, \qquad G^{\mathrm{t}}u - (b - z)v = 0 \tag{7}$$

have nonzero solutions for $v \in \mathbb{C}^r$, $u \in \mathbb{C}^n$. There are r points, denoted by $x_1, \dots, x_r$, at which $w = \infty$, $u = 0$, and v is an eigenvector of b; and there are n points $y_1, \dots, y_n$ at which $z = \infty$, $v = 0$, and u is an eigenvector of a.

The spectral curve is given for both systems by the vanishing of

$$\det\left[\mathbb{M} - \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}\right] = 0, \quad \text{where } \mathbb{M} = \begin{pmatrix} a & -F \\ -G^{\mathrm{t}} & b \end{pmatrix}$$

and the solutions to the linear equations embed Σ in $\mathbb{CP}_{n+r-1}$.

Proposition 2.1. *The genus of* Σ *is* $g = (n-1)(r-1)$.

This follows by equating $2g-2$ to the number of zeros of the 1-form α introduced below in Eq. (15); all the zeros are at $w = \infty$ or $z = \infty$.

3. Line bundles

The second ingredient in the construction is the line bundle $L \to \Sigma$, defined to be the dual of the tautological bundle on $\mathbb{CP}_{n+r-1}$, restricted to Σ. The curve determines the eigenvalues of a matrix at each point, and the fibres of the line bundle are dual to the corresponding eigenspaces—that is, L is dual to the line bundle given by the solutions to the linear system (7) at each point of Σ.

Proposition 3.1. *For generic* F *and* G, $\deg(L) = nr$.

Proof. Let $\tilde{F}, \tilde{G}, \tilde{b}$ denote the matrices obtained by deleting the first column from F, G, and the first row and column from b. Let $\tilde{L} \to \tilde{\Sigma}$ be the line bundle and curve determined by $a, \tilde{b}, \tilde{F}, \tilde{G}$, and put $\delta = \deg(L)$ and $\tilde{\delta} = \deg(\tilde{L})$. The map that sends (u, v) to the first component of v is a holomorphic section of L, and so has δ zeros. Of these, $r-1$ are at $w = \infty$, and n are at $z = \infty$. The remainder are at the finite values of w and z at which there are nonzero solutions to the linear equations

$$(a-w)\tilde{u} - \tilde{F}\tilde{v} = 0, \qquad \tilde{G}^{\mathrm{t}}u - (\tilde{b} - z)\tilde{v} = 0, \qquad G^{1\mathrm{t}}\tilde{u} = 0, \tag{8}$$

in other words they are given by the finite zeros of the holomorphic section $(\tilde{v}, \tilde{u}) \mapsto \tilde{G}^{\mathrm{t}}\tilde{u}$ of $\tilde{L}$. Such a section has $r-1$ zeros at $w = \infty$ and is nonvanishing (for generic G) at $z = \infty$. It therefore has $\tilde{\delta} - r + 1$ zeros at finite values of w and z. Hence

$$\delta - (r-1) - n = \tilde{\delta} - (r-1)$$

which gives $\delta = \tilde{\delta} + n$. For $r = 1$, the only points of Σ at which $v_1 = 0$ are the n points above $z = \infty$, so the result follows by induction on r. □

For generic matrices, $H^0(\Sigma, L)$ has dimension $nr - g + 1 = n + r$, with the global sections of L determined by constant row vectors of length $n + r$.

Let us put

$$L_1 = L \otimes L_{y_1}^{-1} \otimes \cdots \otimes L_{y_n}^{-1}, \qquad L_2 = L \otimes L_{x_1}^{-1} \otimes \cdots \otimes L_{x_r}^{-1}$$

and denote by π_1 and π_2 the projections from Σ onto the w- and z-spheres. Then L_1 and L_2 have respective degree $n(r-1)$ and $r(n-1)$, and

$$\deg(\pi_{1*}L_i) = \deg(L_i) + 1 - g - \deg(\pi_i) = 0.$$

Proposition 3.2. *The direct images of the bundles L_1 and L_2 on $\mathbb{CP}_1$ given by the two projections above are degree 0 vector bundles* (*and hence generically trivial*) *of rank r and n*, *respectively*.

The global sections of $\pi_{1*}L_1$ and $\pi_{2*}L_2$ are given by constant row vectors in $\mathbb{C}^r$ and $\mathbb{C}^n$, respectively. In the same way as in [7], multiplication of the corresponding sections of L_1 and L_2 over Σ by z and w, respectively, determines two meromorphic matrix-valued functions $Z(w)$ and $W(z)$. The original linear system of linear differential equations and its dual are

$$\frac{\mathrm{d}v}{\mathrm{d}w} = Z(w)v \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}z} = -W(z)u.$$

4. Infinitesimal deformations

An infinitesimal isomonodromic deformation of a linear operator of the form (2) is given by making an infinitesimal 'singular gauge transformation' by $\Omega(w)$, where Ω is a matrix-valued rational function of w, with poles at the singularities of the operator. It must be chosen so that the transformed operator has the same singularity structure as the original [9].

Our starting system is equivalent to

$$(a-w)u - Fv = 0, \qquad -G^{\mathrm{t}}u + (b - \partial_w)v = 0. \tag{9}$$

An infinitesimal deformation that changes the eigenvalues at $w = \infty$ will be given by $v \mapsto (1+wD)v$ where $D = \mathrm{diag}(d_1, \ldots, d_r)$ is a small diagonal matrix. The deformed system is

$$(a-w)u - F(1+wD)v = 0, \qquad -G^{\mathrm{t}}u + (b-\partial_w)(1+wD)v = 0.$$

On ignoring terms of order D^2, the second equation is equivalent to

$$\partial_w v - bv + Dv + (1-wD)G^{\mathrm{t}}u = 0.$$

Now

$$\begin{aligned}(1-wD)G^{\mathrm{t}}u &= (1-wD)G^{\mathrm{t}}(a-w)^{-1}F(1+wD)v \\ &= -[D, G^{\mathrm{t}}F]v + G^{\mathrm{t}}(a-w)^{-1}Fv - DG^{\mathrm{t}}a(a-w)^{-1}Fv \\ &\quad + G^{\mathrm{t}}a(a-w)^{-1}FDv,\end{aligned}$$

where we have used $w(a-w)^{-1} = a(a-w)^{-1} - 1$, and ignored terms involving D^2. Therefore, the deformed system is equivalent to (9), but with a, b, F, G changed by

$$\delta a = 0, \qquad \delta b = -D - [D, G^{\mathrm{t}}F], \qquad \delta G^{\mathrm{t}} = -DG^{\mathrm{t}}a, \qquad \delta F = aFD.$$

Equivalently, to the first order in D, $\mathbb{M}$ is deformed to

$$\begin{pmatrix} 1 & 0 \\ DG^{\mathrm{t}} & 1 \end{pmatrix} \left[\mathbb{M} - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right] \begin{pmatrix} 1 & -FD \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Generally, however, the new matrix $b + \delta b$ is not diagonal. To get back to a system of the original form, we make a further constant gauge transformation by C, where

$$C_{ij} = \frac{(G^{\mathrm{t}}F)_{ij}(d_i - d_j)}{b_i - b_j}, \quad i \neq j \tag{11}$$

(note that we require that the b_i's should be distinct). The net result is to replace $\mathbb{M}$ by

$$\mathbb{M}' = \begin{pmatrix} 1 & 0 \\ DG^{\mathrm{t}} & 1 - C \end{pmatrix} \left[\mathbb{M} - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right] \begin{pmatrix} 1 & -FD \\ 0 & 1 + C \end{pmatrix}. \tag{12}$$

If instead we take $\Omega = G^{\mathrm{t}}D(a-w)^{-1}F$, where D is now a diagonal $n \times n$ matrix, then by a similar calculation, we get that (9) is changed by

$$\delta a = D + [FG^{\mathrm{t}}, D], \qquad \delta b = 0, \qquad \delta F = -DFb, \qquad \delta G^{\mathrm{t}} = bG^{\mathrm{t}}D.$$

On rediagonalising a, this gives

$$\mathbb{M}' = \begin{pmatrix} 1 + C & DF \\ 0 & 1 \end{pmatrix} \left[\mathbb{M} - \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 - C & 0 \\ -G^{\mathrm{t}}D & 1 \end{pmatrix}, \tag{13}$$

where now

$$C_{ij} = \frac{(FG^{\mathrm{t}})_{ij}(d_i - d_j)}{a_i - a_j}.$$

This is an isomonodromic deformation in which the positions of the poles at $w = a_i$ move to the eigenvalues of $a + \delta a$, without changing the b_i's.

By comparing (12) and (13), it is clear that isomonodromic deformations of the original system are also isomonodromic deformations of its dual.

5. Elementary deformations

We say that a deformation is *elementary* whenever D has rank one, so that either it changes just one eigenvalue at infinity, or moves just one of the regular singularities. What is the effect on L and Σ of an elementary deformation of the form (13), where $D_{11} = \epsilon$, with the other entries zero?

We shall denote the deformed curve and line bundle by $L' \to \Sigma'$, and use a prime to indicate the quantities associated with them. To the first order in D, we have

$$\mathbb{M}' - \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1+S & DF \\ 0 & 1 \end{pmatrix} \left[\mathbb{M} - \begin{pmatrix} w+D & 0 \\ 0 & z \end{pmatrix} \right] \begin{pmatrix} 1-S & 0 \\ -G^{\mathrm{t}}D & 1 \end{pmatrix},$$

where $S = C + wD$. So we obtain Σ' by replacing a by $a - D$ in $\mathbb{M}$ (this is true only to the first order—that is, only for *infinitesimal* deformations).

Now consider the section σ of L given by $(u, v) \mapsto \nu(u)$, where ν is a constant row vector with 1 in the first entry, and 0's in the other entries. Then $\sigma = 0$ at $x_1, \ldots, x_r, y_2, \ldots, y_n$, where Σ and Σ' intersect. Denote its other zeros by $q_1, \ldots, q_g$. At each such point q, we have $Du = 0$ and

$$\mathbb{M}' \begin{pmatrix} (1+C)u \\ v \end{pmatrix} = \begin{pmatrix} w(1+C)u \\ zv \end{pmatrix}.$$

It follows that Σ and Σ' also intersect at q, and that

$$\sigma' : (u', v') \to \nu(1-C)u'$$

is a section of L' that vanishes at each q_i. We also have that $\sigma' = 0$ at $x_1, \ldots, x_r$.

To understand what happens to σ' at $y_2, \ldots, y_n$, we consider the behaviour of u, v near $z = \infty$, $w = a_i$ $(i \neq 1)$. By expanding in powers of z, we have

$$u = u_0 + z^{-1}u_1 + \mathrm{O}(z^{-2}), \quad v = z^{-1}v_1 + \mathrm{O}(z^{-2}), \quad w = a_i + z^{-1}c_i + \mathrm{O}(z^{-2}),$$

where we take u_0 to have a 1 in the ith entry, with the other entries zero. On substituting into (7), we obtain

$$au_1 - Fv_1 = a_i u_1 + c_i u_0, \qquad G^{\mathrm{t}}u_0 = v_1$$

and hence that

$$(a - a_i I)u_1 = (H + c_i I)u_0,$$

where $H = FG^{\mathrm{t}}$. Since $\nu(u_0) = 0$, it follows that $\nu(u_1) = H_{1i}/(a_1 - a_i)$.

Now take $z^{-1} = \epsilon$. Since $H' = H + \mathrm{O}(\epsilon)$ and $a' = a + \mathrm{O}(\epsilon)$, we have

$$\nu(u') = \nu((1-C)u') = \nu((1-C)u_0) + \frac{\epsilon H_{1i}}{a_1 - a_i} = \mathrm{O}(\epsilon^2).$$

We can conclude that the $n-1$ zeros of σ at $y_2, \ldots, y_n$ are shifted to zeros of σ' at the nearby points on Σ' at which $z = 1/\epsilon$. Therefore, to the first order in ϵ, $\sigma'/(1 - \epsilon z)$ is a meromorphic section of L' with the following properties:

- It has zeros at $q_1, \ldots, q_g$, at $x_1, \ldots, x_r$, and at $y_2, \ldots, y_n$ at all of which Σ and Σ' intersect;
- It has a zero at $z = \infty$, $w = a_1 + \epsilon$, and a pole at the nearby point at which $z = 1/\epsilon$.

Otherwise it is holomorphic and nonzero. So we have the following lemma.

Lemma 5.1. *The line bundle $L \to \Sigma$ has a divisor contained in the intersection of Σ and Σ'. The deformed bundle L' is the bundle over Σ' with the same divisor, but twisted by* $\exp(1-\epsilon z)$ *at* $z=\infty$, $w=a_1+\epsilon$.

This can be said in a simpler way. Let V be an open set in Σ made up of n punctured disks, with centres at the n points at which $z=\infty$. We use z as a coordinate on V.

Suppose that we are given a divisor

$$D=\sum k_i m_i, \quad m_i \in \Sigma, \ \ k_i \in \mathbb{Z},$$

together with a nonvanishing holomorphic function $P(z)$, defined on V. Then we have a line bundle on $L_\Sigma \to \Sigma$, given by tensoring L_D by the bundle with transition function P. If we have similar nearby objects D' (a divisor on Σ') and P', then we can characterise the change from L_Σ to $L_{\Sigma'}$ by giving the change $\delta D=\sum k_i\,\delta m_i$ in D and the change δP in P as a function of z. Of course, there is a lot of redundancy because different choices of D and P will give the same bundle L_Σ.

For each $\Sigma \in \mathcal{P}$, let E be the degree-zero line bundle with transition function $P=\exp(wz)$, where w is defined as a function of z by restriction to Σ. Then $L\otimes E$ is determined by P and the divisor

$$D=q_1+\cdots+q_g+x_1+\cdots+x_r+y_2+\cdots+y_n. \tag{14}$$

The lemma can be restated as follows.

Lemma 5.2. *The deformation of $L\otimes E$ is given by* $\delta D=0$, $\delta P=0$.

Lemma 5.1 is also true for an elementary deformation which changes b_1, except that the twist is by $\exp(1-\epsilon w)$ at $w=\infty$, $z=b_1+\epsilon$

6. Symplectic approach

The data of a dual pair of linear operators are encoded in the following:

- a curve Σ of genus $(n-1)(r-1)$, the spectral curve of the system;
- an embedding of Σ into $\mathbb{CP}_1\times\mathbb{CP}_1$, with the projections onto the two $\mathbb{CP}_1$'s of degree r and n, respectively, and
- a line bundle $L\to\Sigma$ of degree nr.

We shall consider how to generate isomonodromic deformations (of both systems) from Hamiltonian flows on a symplectic manifold $\mathcal{M}$ constructed, following ideas in [1], from curves in $\mathbb{CP}_1\times\mathbb{CP}_1$.

Let $M=\mathbb{CP}_1\times\mathbb{CP}_1\setminus(\infty,\infty)$ (our spectral curves do not pass through the excluded point, and are therefore embedded in M). Let $\omega=\mathrm{d}w\wedge\mathrm{d}z$: this is a meromorphic 2-form on M, and it is holomorphic and symplectic except where $w=\infty$ or $z=\infty$.

Let $\mathcal{P}$ denote the space of curves in M with the properties above (in fact we shall consider only local deformations, so $\mathcal{P}$ should be thought of as an open neighbourhood of a given spectral curve). Then $\dim(\mathcal{P}) = nr + n + r$. This can be seen by representing $\Sigma \in \mathcal{P}$ as the zero set of a polynomial $p(w, z)$ of degree n in w and r in z. The polynomial is determined by Σ up to scale. Alternatively, if Z is a local section of $TM|_{\Sigma}$, then the restriction of $i_Z\omega$ to Σ is a meromorphic section of the canonical bundle with poles of order at most 2 at $w = \infty$ and $z = \infty$. It vanishes whenever Z is tangent to Σ. Therefore, the normal bundle is

$$N = K \otimes \pi_1^*(\mathcal{O}(2)) \otimes \pi_2^*(\mathcal{O}(2)),$$

which has degree $2g - 2 + 2n + 2r = g - 1 + nr + n + r$.

We remark that

$$\alpha = \frac{\mathrm{d}w}{\partial p/\partial z} = -\frac{\mathrm{d}z}{\partial p/\partial w} \tag{15}$$

is a natural holomorphic 1-form on Σ, with zeros only at the points at infinity; by considering the orders of these, we can compute the genus of Σ.

The manifold $\mathcal{M}$ is obtained from $\mathcal{P}$ by attaching an 'extended Jacobian' to each curve. A point of $\mathcal{M}$ is a point on the Jacobian of one of the curves, together with a frame for the line bundle defined up to the second order at each point at infinity, modulo an overall scale. In other words, a point of $\mathcal{M}$ is represented by a triple (Σ, B, s) where

- $\Sigma \in \mathcal{P}$;
- $B \to \Sigma$ is a degree zero line bundle;
- s is a nonvanishing section of B on a neighbourhood of the $n + r$ points of Σ at which ω is singular.

Two such triples (Σ, B, s) and (Σ', B', s') determine the same point of $\mathcal{M}$ whenever $\Sigma = \Sigma'$, $B = B'$ and $s - ks'$ has zeros of order 2 at each of the $n + r$ points at infinity, for some constant k. The data in the frames add $2(n + r) - 1$ extra dimensions, so

$$\dim(\mathcal{M}) = \dim(\mathcal{P}) + g + 2(n + r) - 1 = 2(nr + n + r).$$

The holomorphic symplectic structure Ω on $\mathcal{M}$ is analogous to that on a cotangent bundle, with the extended Jacobians playing the role of the fibres. The symplectic form is the exterior derivative of a 'canonical 1-form' Θ, which is defined as follows.

Let T be a tangent vector to $\mathcal{M}$ at (Σ, B, s) and let Z be its projection into $\mathcal{P}$. That is, Z is a section of N, so the restriction of $i_Z\omega$ to Σ is a well-defined meromorphic 1-form, with poles at the points at infinity. Suppose that s is defined on an open set V (not necessarily connected) containing the points at infinity. Let $U \subset \Sigma$ be a second open set in the complement of the points at infinity such that V and U cover Σ; and let β be a meromorphic section of $B|_U$ with equal numbers of poles p_i and zeros q_i, none of which are at infinity. We define

$$i_T\Theta = \sum \int_{q_i}^{p_i} i_Z\omega - \frac{1}{2\pi i} \sum \oint \log\left(\frac{\beta}{s}\right) i_Z\omega, \tag{16}$$

where the integrals on the right are around contours in $V \cap U$ surrounding the points at infinity. Given β, $i_T\Theta$ is well defined up to the addition of terms of the form

$$\oint i_Z\omega = i_T\mathrm{d}\left(\oint \theta\right), \tag{17}$$

where $\theta = w\,\mathrm{d}z$ and the integrals are around a closed contour in U. We could, for example, take β to be a meromorphic section, so that $B = \sum(q - p)$; but for a general choice, β might be highly singular at infinity.

If β is replaced by $\beta' = m\beta$, where m is meromorphic on U with zeros at the poles of β, then the $i_Z\Theta$ is unchanged up to the freedom above. This follows by applying the following with $\gamma = i_Z\omega$.

Lemma 6.1. *Let $U \subset \Sigma$ be a connected open set with boundary ∂U made up of closed contours. Let m be a meromorphic function on U with equal number of zeros and poles; and let γ be a holomorphic 1-form on U. Then* $\log(m)$ *can be defined on ∂U and, modulo integral multiples of the periods of γ,*

$$\frac{1}{2\pi i}\oint_{\partial U} \log(m)\gamma = \sum \int \gamma,$$

where on the right, the sum is over pairs of poles and zeros of m, and the integrals are along paths in U from the zero to the pole in each pair.

By the 'periods of γ', we mean the integrals of γ around closed contours in U. The proof is by extending $\log(m)$ to the complement of a set of cuts along closed paths on Σ and along paths joining paired poles and zeros of m.

It follows that $\Omega = \mathrm{d}\Theta$ is a well-defined 2-form on $\mathcal{M}$. It is given explicitly as follows. We choose β at each $m \in \mathcal{M}$ in a neighbourhood of a given point. Then the points of $\mathcal{M}$ can be labelled by Σ, the zeros and poles q_i and p_i of β, and the function β/s defined in an annulus around each point at infinity. We use the coordinate w to identify the annuli around $w = \infty$ on neighbouring curves; and z for those around $z = \infty$. Then a tangent to $\mathcal{M}$ is represented by a tangent vector Z to M at each of the points q, p, a vector field, also denoted by Z, connecting Σ to the nearby curve in a neighbourhood of each point at infinity, and the variation in $\log(\beta/s)$, as a function of w or z.

Proposition 6.2. *The 2-form $\Omega = \mathrm{d}\Theta$ on $\mathcal{M}$ is a closed nondegenerate 2-form given by the following expression*

$$\Omega(T, T') = \sum_p \omega_p(Z, Z') - \sum_q \omega_q(Z, Z') + \sum_{w,z=\infty} \oint (g' i_Z\omega - g i_{Z'}\omega). \tag{18}$$

Note that the right-hand side vanishes identically whenever $Z = 0$ and g is constant, so Ω is well defined on $\mathcal{M}$ (it descends under the quotient by constant scaling of s).

7. The Hamiltonian system

We shall construct a Hamiltonian on $\mathcal{M}$ for each point at infinity on Σ which generates an isomonodromic deformations of Harnad's dual systems. First we deal with the Hamiltonian associated with y_1.

For each (Σ, B, s), we choose a square root of $B \otimes K$. The choices are parameterised by $H^1(\Sigma, \mathbb{Z}_2)$, and can be made continuously as Σ and B vary. In general, since its degree is $g-1$, the line bundle $(B \otimes K)^{1/2} \otimes E^{-1}$ has a section τ which is holomorphic except for a simple pole at y_1. This section is unique up to scale. Denote the zeros of τ by $q_1, \dots, q_g$ and denote by μ the meromorphic 1-form on Σ which has zeros at the points q_i and a double pole at y_1. Again with the qualification 'in general', this exists and is unique up to scale, which we fix by requiring that $\mu - \mathrm{d}z$ should be holomorphic (note that the residue of μ at y_1 necessarily vanishes).

The quotient τ^2/μ is a meromorphic section of $B \otimes E^{-2}$. It has simple poles at the points q_i and is otherwise holomorphic. From it, we obtain a section β of B (unique up to scale) with the following properties:

- It is holomorphic except for simple poles at the points q_i, and for an essential singularity at y_1, where $\exp(2wz)\beta$ is holomorphic.
- Its zeros are the other zeros $p_1, \dots, p_g$ of μ.

We define the function h on $\mathcal{M}$

$$h = \frac{1}{2\pi i} \oint_{y_1} \log\left(\frac{\beta}{s}\right) \mathrm{d}z, \tag{19}$$

where the integral is around a contour surrounding y_1.

We shall calculate the derivative of h along a tangent T' to $\mathcal{M}$. First suppose that T' does not move Σ. Put $H = \exp(2wz)\beta/s$. Then H is holomorphic at y_1 and

$$T'(h) = i_{T'}\,\mathrm{d}\left(\frac{1}{2\pi i}\oint_{y_1} \log(H)\,\mathrm{d}z\right) = i_{T'}\,\mathrm{d}\left(\frac{1}{2\pi i}\oint_{y_1} \log(H)\mu\right) = \frac{1}{2\pi i}\sum \oint_{y_i} g'\mu,$$

where g' is the change in $\log(\beta/s)$, which in this case is holomorphic at $z=\infty$. If T' moves Σ, but leaves β/s unchanged, then $T'(h) = 0$. We conclude that the Hamiltonian vector field $T = T_h$ generated by h is given in the representation of the previous section by taking

$$i_Z\omega|_\Sigma = \mu, \quad g = 0.$$

How does h generate isomonodromic deformations? We associate a dual pair with a point of $\mathcal{M}$ by identifying $L \otimes E$ with

$$(B \otimes K)^{1/2} \otimes L_{x_1} \otimes \cdots \otimes L_{x_r} \otimes \cdots \otimes L_{y_1} \otimes \cdots \otimes L_{y_n}.$$

In the notation of Section 5, $L \otimes E$ is given by the divisor D in (14) together with a transition function P, defined by $P^2 = \beta/s$; and the Hamiltonian flow gives $\delta D = 0, \delta P = 0$, which is isomonodromic by Lemma 5.2. With this identification, therefore, h generates isomonodromic deformations of the dual pair of linear operators determined by L. The deformation is elementary; it changes a_1, leaving $a_2, \dots, a_n, b_1, \dots, b_r$ fixed.

If we relabel $h, F, \beta, \dots$ as h_1, F_1, β_1, and so on, and use subscripts to denote the analogous quantities defined with a_1 replaced by a_i. The Hamiltonians h_i generate deformations that move the other a_i, leaving b_i fixed.

Proposition 7.1. *The Hamiltonians h_i, $i = 1, \dots, h_n$ are in involution.*

Proof. We have to show that $T_i(h_j) = 0$, where T_i is Hamiltonian vector field generated by h_i. Now

$$\begin{aligned} T_i(h_j) &= i_{T_i}\left(\frac{1}{2\pi i}\oint_{y_j} \log(F_j)\,\mathrm{d}z\right) = i_{T_i}\left(\frac{1}{2\pi i}\oint_{y_j} \log(F_i) + \log\left(\frac{\beta_j}{\beta_i}\right)\mathrm{d}z\right) \\ &= i_{T_i}\left(\frac{1}{2\pi i}\oint_{y_j} \log\left(\frac{\beta_j}{\beta_i}\right) i_{Z_j}\omega\right) = i_{T_i}\left(\sum_k \frac{1}{2\pi i}\oint_{y_k} \log\left(\frac{\beta_j}{\beta_i}\right) i_{Z_j}\omega\right) \\ &= \sum_{\text{poles}} \omega(Z_i, Z_j) - \sum_{\text{zeros}} \omega(Z_i, Z_j) = 0, \end{aligned}$$

where in the penultimate line, the sums are over the zeros and poles of β_j/β_i. In going from the fourth to the fifth line, we use the fact that β_j/β_i is holomorphic at $z = \infty$, and that the restriction of $i_{Z_j}\omega$ to Σ is nonsingular except at y_j. The last line follows because either Z_i or Z_j vanishes at each pole and zero. □

By interchanging the roles of w by $-z$, and z by w, we similarly define Hamiltonians k_i $(i = 1, \dots, r)$ that generate the other isomonodromic flows; a direct extension of the proof above shows that these are involution with each other and with the h_i's.

One could recover the nonautonomous picture of Harnad's original paper [6] by ignoring the bundle E. One would have to construct two sets of commuting Hamiltonians and perform some symplectic quotients. A nice byproduct of this approach is an explicit symplectic isomorphism between Harnad's space of $n \times r$ matrices and a symplectic quotient of $\mathcal{M}$. For details, see [10].

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